

The effect of non-Gaussian curvature perturbations on the formation of primordial black holes

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Abstract

This paper explores the consequences of non-Gaussian cosmological perturbations for the formation of primordial black holes (PBHs). A non-Gaussian probability distribution function (PDF) of curvature perturbations is presented with an explicit contribution from the three-point correlation function to linear order. The consequences of this non-Gaussian PDF for the large perturbations that form PBHs are then studied. Using the observational limits for the non-Gaussian parameter f_{NL} , new bounds to the mean amplitude of curvature perturbations are derived in the range of scales relevant for PBH formation.

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I. INTRODUCTION

The non-linear theory of perturbations is an important part of the study of the early universe because only at this level of precision one can distinguish specific models of inflation and even rule out some of them through observations of the correspondent non-Gaussian statistics. The possibility of observing non-Gaussian correlations in the temperature fluctuations of the cosmic microwave background radiation (CMB) [1] and in the count of large scale structure (LSS)[2, 3, 4] has fostered the study of non-linear perturbations for distinct models of the origin of structure (see e.g. Refs. [5, 6, 7], and see Ref. [8] for an extended review of both observational and theoretical aspects of non-Gaussianity,).

The properties of the early universe imprinted in the observed perturbations are better studied when we make use of the curvature perturbation $\mathcal{R}(t, \mathbf{x})$ on comoving hypersurfaces [55]. This quantity is defined in a gauge-invariant way and, when sourced by adiabatic matter perturbations, the growing perturbation mode remains constant on scales larger than the cosmological horizon H^{-1} , where $H = d \ln(a)/dt$ is the Hubble parameter [9, 10].

A fundamental observable of cosmological perturbations is the mean amplitude. The classical amplitude of metric perturbations \mathcal{R}_{cl} is derived by solving the perturbed Einstein equations to linear order. Statistically this classical amplitude is written in terms of the two point correlation function as [11],

$$\langle \mathcal{R}_{\text{G}}(\mathbf{k}_1) \mathcal{R}_{\text{G}}(\mathbf{k}_2) \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) |\mathcal{R}_{\text{cl}}(k)|^2, \quad (1)$$

where $\mathcal{R}_{\text{G}}(\mathbf{k})$ are Gaussian perturbations in Fourier space, and \mathbf{k}_1 and \mathbf{k}_2 are vectorial Fourier momenta. The two-point correlator defines also the dimensionless power-spectrum

[55] In the longitudinal gauge, the scalar isotropic curvature perturbation or Bardeen potential Φ modifies the Friedmann background metric to the form:

$$ds^2 = -[1 + 2\Phi(t, \mathbf{x})] dt^2 + a(t)^2 [1 - 2\Phi(t, \mathbf{x})] d\mathbf{x}^2.$$

where t is the cosmic time, $a(t)$ is the scale factor and $d\mathbf{x}$ is the comoving spatial part. The comoving curvature perturbation $\mathcal{R}(t, \mathbf{x})$ is defined in the comoving gauge from the metric

$$ds^2 = -N^2(t, \mathbf{x}) dt^2 + a(t)^2 e^{2\mathcal{R}(t, \mathbf{x})} \delta_{ij} (dx^i + N^i(t, \mathbf{x}) dt) (dx^j + N^j(t, \mathbf{x}) dt),$$

where δ_{ij} is the Kronecker delta, and N and N^i are functions algebraically determined from the symmetries of the perturbed equations of motion.

$\mathcal{P}(k)$ as

$$\langle \mathcal{R}_G(\mathbf{k}_1) \mathcal{R}_G(\mathbf{k}_2) \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) \frac{2\pi^2}{k_1^3} \mathcal{P}(k_1). \quad (2)$$

The power-spectrum encodes important information of the underlying cosmological model. For example, for perturbations generated from a single inflationary field ϕ with a potential V dominating the cosmological dynamics, $\mathcal{R}_G(k)$ is a random field of perturbations about the quasi-de Sitter background, and the power-spectrum is given by [12]

$$\mathcal{P}(k) = \frac{H_*^4}{(2\pi)^2 \dot{\phi}_*^2 m_{\text{Pl}}^2} \approx \frac{V_*^3}{(dV/d\phi)_*^2 m_{\text{Pl}}^2}, \quad (3)$$

where $m_{\text{Pl}} = 8\pi G$ is the Planck mass and where an asterisk denotes evaluation at t_* , the time when the relevant perturbation mode exits the cosmological horizon, $k = a_* H_* = a(t_*) H(t_*)$.

The tilt of the power-spectrum is parametrised with a second observable, the spectral index which is defined as

$$n_s = \frac{d}{d \ln k} \ln \mathcal{P}(k). \quad (4)$$

With this definition, $n_s < 0$ means that the power-spectrum is larger for large scales, in such case we have a red spectrum. Equivalently $n_s > 0$ corresponds to a larger power for small scales and this is called a blue spectrum.

The power-spectrum and the tilt are derived directly from linear perturbations. Assuming linearity for the perturbations of the CMB, it is possible to determine with great accuracy the numerical values of the power-spectrum and its tilt on scales larger than the horizon at the time of last scattering. At such scales, the most recent observations of the CMB and the galaxy count give $\mathcal{P} = 2.4 \times 10^{-9}$, $n_s = -0.05 \pm 0.01$ [1]. Any successful theory of structure formation must meet these values at the relevant scales. Higher order correlations of the perturbation field \mathcal{R} offer an exciting way to distinguish between these cosmological models.

The deviations from Gaussianity are described to lowest order by the non-linear parameter f_{NL} . Mathematically, this parameter appears in the expansion [13]

$$\mathcal{R}(k) = \mathcal{R}_G(k) - \frac{3}{5} f_{\text{NL}} (\mathcal{R}_G \star \mathcal{R}_G(k) - \langle \mathcal{R}_G^2 \rangle), \quad (5)$$

where a star denotes a convolution of two copies of the field. We use this definition throughout the paper. At the present time, the precision of the observations can only provide limits

for f_{NL} . The WMAP satellite gives the constraints $-54 < f_{\text{NL}} < 114$ at the 95% confidence level [1].

When one computes second-order perturbations to the Einstein equations, higher order correlations become non-vanishing (see e.g. Refs. [14, 15]). Alternatively such correlator can be obtained from the third-order quantum perturbations to the Einstein-Hilbert action (Pioneering works using this method are Refs. [16, 17, 18]). The three-point correlation function in Fourier space at the tree level can be directly computed from the definitions in Eqs. (1)(2) and (5) to find [13]

$$\langle \mathcal{R}(k_1)\mathcal{R}(k_2)\mathcal{R}(k_3) \rangle = - (2\pi)^3 \delta \left(\sum_i \mathbf{k}_i \right) 4\pi^4 \frac{6}{5} f_{\text{NL}} \left[\frac{\mathcal{P}(k_1)\mathcal{P}(k_2)}{k_1^3 k_2^3} + 2 \text{ perm} \right]. \quad (6)$$

Non-linear correlations have been explored not only in the CMB anisotropies [1, 19], but also in counts of galaxies and clusters [2, 4, 20, 21]. The latter deserve special mention. The distribution of galaxies and clusters account for density perturbations that formed such structures in the matter-dominated era. The statistics of these objects gives us extra information about the primordial non-Gaussianities in the universe. However, in studying the formation of galaxies and clusters from the seed perturbations, one must be careful in the treatment of intermediate stages because the growth of perturbations after horizon entry is intrinsically non-linear. This may blur the primordial non-Gaussian statistical effects (For a review, see Ref. [3]).

In this paper we study non-Gaussianity for a range of wavelengths not covered by the observations mentioned above by looking at the formation of Primordial Black Holes (PBHs). PBHs were formed during the radiation era of our universe from large amplitude perturbations. The importance of non-Gaussianity in this context is that a small deviation from linearity would change the probability of formation significantly. The effects of non-Gaussian perturbations on PBHs have been studied for specific models [22, 23, 24] but a precise quantification of the non-Gaussian effects is still required and it is only now, with a much better understanding of the effects of higher order perturbations, that we are able to describe the general effects on PBHs. The discussion of non-Gaussian effects on PBH formation turns crucial in the light of recent works (e.g. [25]) which claim that only through exotic extensions of the canonical slow-roll inflationary potentials can produce considerable amounts of PBHs. Here we explore whether non-Gaussian perturbations could ease this difficulty.

To look at the effects of non-Gaussianity we make use of a non-Gaussian PDF for cur-

vature perturbations with a direct dependence on the non-linear parameter f_{NL} as derived in [26]. We adapt this PDF to calculate probabilities for perturbations relevant to PBH formation. We also explore the amount of non-Gaussianity that is allowed from the bounds to the total mass fraction of PBHs. In principle, the PBH abundance could be used to probe non-Gaussianity on scales not reached through CMB or LSS observations. We find that even though the possible limits on f_{NL} in the relevant mass regime are weak, its current allowed values in the CMB scale can be used to generate new bounds on the mean amplitude of curvature perturbations \mathcal{R} for the wavelength range $10^{-16} < \lambda/Mpc < 1$.

This paper is organised as follows. In Section II we construct the appropriate non-Gaussian probability distribution function (PDF) to calculate the probability of non-linear perturbations forming PBHs. In Section III we characterise the effects of the non-Gaussian PDF on the probability of PBH formation. We recover the results presented in previous studies [22, 23] and reconcile the discrepancy in their conclusions. In Section IV, we make use of the Press-Schechter formalism to derive the abundance of PBHs including non-Gaussian effects. We show how the constraint on the amplitude of perturbations is modified by the introduction of a non-Gaussian mass function and report on new bounds on the variance of curvature perturbations. We conclude in Section V with a summary of results.

II. THE NON-GAUSSIAN PDF

In linear perturbation theory one makes use of the central limit theorem to construct the probability distribution function (PDF). To first order, the perturbation modes are independent of each other. If we define the field of perturbations \mathcal{R} with zero spatial average, then the central limit theorem indicates that the PDF of \mathcal{R} is a normal distribution dependent on a single parameter, the variance $\langle \mathcal{R}^2 \rangle$,

$$\mathbb{P}_{\text{G}}(\mathcal{R}) \approx \frac{1}{\sqrt{\langle \mathcal{R}^2 \rangle}} \exp \left(-\frac{\mathcal{R}^2}{2\langle \mathcal{R}^2 \rangle} \right). \quad (7)$$

An immediate effect of including the non-linearity of the perturbations is the interaction of distinct perturbation modes. Indeed, as shown in Eq. (5), the parametrisation of higher order contributions to \mathcal{R} is described in terms of convolutions in Fourier space. The requirements of the central limit theorem are not met, so it can no longer be used to construct the PDF.

In a recent paper a first-order correction to the Gaussian PDF was explored, and a new PDF was derived, which includes the linear order contribution from the 3-point function. In the following we describe the elements of such derivation.

As is customary in the treatment of perturbations, we first smooth the field over a given mass scale k_M with a window function $W_M(k)$,

$$\bar{\mathcal{R}}(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^3} W_M(k) \mathcal{R}(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x})}. \quad (8)$$

Here we use a truncated Gaussian window function:

$$W_M(k) = \Theta(k_{\max} - k) \exp\left(-\frac{k^2}{k_M^2}\right), \quad (9)$$

where Θ is the Heaviside function and the fiducial scale k_{\max} is introduced to avoid ultraviolet divergences. The amplitude of the perturbation is parametrised by the central value of the configuration $\bar{\mathcal{R}}(\mathbf{x} = 0) = \zeta_{\mathcal{R}}$. This parametrisation is particularly useful when we pick the relevant perturbations for the formation of PBHs [27]. The non-Gaussian probability distribution function for a perturbation with central amplitude $\zeta_{\mathcal{R}}$ is [26]

$$\mathbb{P}_{\text{NG}}(\zeta_{\mathcal{R}}) = \mathbb{P}_{\text{G}}(\zeta_{\mathcal{R}}) \left[1 + \left(\frac{\zeta_{\mathcal{R}}^3}{\Sigma_{\mathcal{R}}^3} - \frac{3\zeta_{\mathcal{R}}}{\Sigma_{\mathcal{R}}} \right) \frac{\mathcal{J}}{\Sigma_{\mathcal{R}}^3} \right], \quad (10)$$

where $\Sigma_{\mathcal{R}}^2$ is the variance of the smoothed field, related to the power-spectrum by

$$\Sigma_{\mathcal{R}}^2(M) = \int \frac{dk}{k} W_M^2(k) \mathcal{P}(k), \quad (11)$$

and the factor \mathcal{J} encodes the non-Gaussian contribution to the PDF:

$$\mathcal{J} = \frac{1}{6} \int \frac{d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3}{(2\pi)^9} W_M(k_1) W_M(k_2) W_M(k_3) \langle \mathcal{R}(\mathbf{k}_1) \mathcal{R}(\mathbf{k}_2) \mathcal{R}(\mathbf{k}_3) \rangle, \quad (12)$$

$$= -\frac{1}{5} \int \frac{d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3}{(4\pi)^2 \prod_i W_M^{-1}(k_i)} \delta\left(\sum_i \mathbf{k}_i\right) f_{\text{NL}} \left[\frac{\mathcal{P}(k_1) \mathcal{P}(k_2)}{k_1 k_2} + 2 \text{ perm.} \right], \quad (13)$$

where the last equation is valid at tree level in the expansion of $\langle \mathcal{R} \mathcal{R} \mathcal{R} \rangle$. This limitation is justified as long as the loop contributions to the three-point function, generated from the convolution of \mathcal{R} -modes, are sub-dominant. Such is the case for $f_{\text{NL}} \leq 1/\mathcal{P}(k)$ for all values of k . As will be seen in section IV, the tested f_{NL} does not exceed such values.

Further details of the derivation of the PDF in Eq. (10) can be found in Ref. [26]. Here is sufficient to say that the time-dependence of this probability is eliminated when

the averaging scale is $k_M \leq a(t)H(t)$ because the growing mode of the perturbation \mathcal{R} is constant on superhorizon scales [9, 10].

We now proceed to integrate the non-Gaussian factor in Eq. (13) over the amplitudes and scales relevant to PBHs. The expression in Eq. (13) is integrated within the limits k_{\min} and k_{\max} defined conveniently to cover the relevant perturbation modes for PBH formation. These objects are formed long before the matter-radiation equality, so in the large-box (small wavenumber) limit of the integral (13) we can choose $k_{\min} = H_0$, the Hubble horizon today. This is a reasonable lower limit for integrating perturbations relevant for PBH formation [28]. At the other end of the spectrum, the smallest PBHs have the size of the Hubble horizon at the end of inflation. We therefore use $k_{\max} = a(t_{\text{END}})H_{\text{END}}$, the comoving horizon at the end of inflation, as a suitable upper limit. It is important to mention that, even though the integral in Eq. (13) should add all k -modes, finite limits are imposed to avoid logarithmic divergences. Moreover, due to the window function $W_M(k)$, the dominant part of the integral is independent of the choice of integration limits as long as they remain finite.

We solve integral (13) for the limit of equilateral configurations of the three-point correlator, that is, correlations for which $k_1 = k_2 = k_3$. This is not merely a computational simplification. In the integral, each perturbation mode has a filter factor $W_M(k)$ which, upon integration, picks dominant contributions from the smoothing scale k_M common to all perturbation modes [56]. We take this argument as an ansatz, in which case \mathcal{J} can be written in the suggestive way:

$$\mathcal{J} = -\frac{1}{8} \int_{k_{\min}}^{k_{\max}} \frac{dk}{k} [W_M(k)\mathcal{P}(k)]^2 \left(\frac{6}{5} f_{\text{NL}} \right). \quad (14)$$

In the following sections we compute \mathcal{J} numerically for inflationary perturbations generated in a single field slow-roll inflationary epoch, and for the case of constant f_{NL} . This will be used to test the effects of non-Gaussianity on PBHs.

[56] This is an important restriction to the kind of non-Gaussianities that can be tested by PBHs [29]. If a strong non-Gaussian signature is encountered exclusively for isosceles triangulations, this cannot affect the formation of PBHs.

III. NON-GAUSSIAN MODIFICATIONS TO THE PROBABILITY OF PBH FORMATION

The simplest models of structure formation within the inflationary paradigm are those where a single scalar field drives an accelerated expansion of the spacetime and its quantum fluctuations give birth to the observed structure in subsequent stages of the universe (For an extended review of the inflationary paradigm, see Ref. [30]). Here we study the curvature perturbations generated at the time of single scalar-field inflation including contributions from non-linear perturbations.

The effects of non-Gaussianity on PBHs have been explored in the past but with inconclusive results. Bullock and Primack [22] studied the formation of PBHs numerically for perturbations with blue spectra ($n_s > 0$) and non-Gaussian contributions. The motivation for this was that any inflationary model with a constant tilt and consistent with the normalisation of perturbations at the CMB scale, must have a blue spectrum to produce a significant number of PBHs [31, 32]. Their analysis is based on the stochastic generation of perturbations on superhorizon scales, together with a Langevin equation for computing the PDF. For all the cases tested, the non-Gaussian PDF is skewed towards small fluctuations, so PBH production, which integrates the high amplitude tail, is suppressed with respect to the Gaussian case. An example of the kind of potentials studied in Ref. [22] is

$$V_1(\phi) = V_0 \begin{cases} 1 + \arctan\left(\frac{\phi}{m_{\text{Pl}}}\right), & \text{for } \phi > 0, \\ 1 + (4 \times 10^{33}) \left(\frac{\phi}{m_{\text{Pl}}}\right)^{21}, & \text{for } \phi < 0. \end{cases} \quad (15)$$

where V_0 is the amplitude of the potential at $\phi = 0$.

Another way of generating large perturbations in the inflationary scenario is to consider localised features in the potential dominating the dynamics. As one can see from Eq. (3), an abrupt change in the potential would generate a spike in the spectrum of perturbations. The effects of non-Gaussianity for an inflationary model producing features in an otherwise red spectrum ($n_s < 0$) were explored by Ivanov [23] using the toy model

$$V_2(\phi) = \begin{cases} \lambda \frac{\phi^4}{4} & \text{for } \phi < \phi_1, \\ A(\phi_2 - \phi) + \lambda \frac{\phi^4}{4} & \text{for } \phi_2 > \phi > \phi_1, \\ \tilde{\lambda} \frac{\phi^4}{4} & \text{for } \phi > \phi_2. \end{cases} \quad (16)$$

where λ and $\tilde{\lambda}$ are coupling constants. Through a stochastic computation of the PDF, it

was found that large amplitude perturbations were more abundant for a non-Gaussian PDF than for a Gaussian one.

To understand this difference and generalise the effects of non-Gaussianity, we look at the fractional difference of the Gaussian and non-Gaussian PDFs:

$$\frac{\mathbb{P}_{\text{NG}} - \mathbb{P}_{\text{G}}}{\mathbb{P}_{\text{G}}} = \left[\left(\frac{\zeta_{\mathcal{R}}^3}{\Sigma_{\mathcal{R}}^3} - 3 \frac{\zeta_{\mathcal{R}}}{\Sigma_{\mathcal{R}}} \right) \frac{\mathcal{J}}{\Sigma_{\mathcal{R}}^3} \right]. \quad (17)$$

Note that both Refs. [22] and [23] use perturbations generated in a piecewise slow-roll inflationary potential for which inflation is controlled by keeping the slow-roll parameters $\epsilon \equiv 1/2(m_{\text{Pl}}V'/V)^2$ and $\eta \equiv m_{\text{Pl}}^2(V''/V)$ smaller than one. Here we make use of the slow-roll approximation to explore the qualitative effects of Eq. (17). To linear order, there is a straightforward expression for the spectral index in terms of these parameters [12],

$$n_s = 2(\eta - 3\epsilon). \quad (18)$$

On the other hand, carrying a first order expansion on slow-roll parameters, Maldacena, provides an expression for the non-linear factor f_{NL} in terms of the slow-roll parameters [16],

$$f_{\text{NL}} = \frac{5}{12} (n_s + \mathcal{F}(k)n_t) = \frac{5}{6} (\eta - 3\epsilon + 2\mathcal{F}(k)\epsilon), \quad (19)$$

where $n_t = 2\epsilon$ is the scalar-tensor perturbation tilt and $\mathcal{F}(k)$ is a number depending on the triangulation used. For the case of equilateral configurations, when $\mathcal{F} = 5/6$,

$$f_{\text{NL}} = \frac{5}{6} \left(\eta - \frac{4}{3}\epsilon \right)_{\text{eq}}. \quad (20)$$

We use this last result to evaluate the integral (14). The non-Gaussian effect on the PDF is illustrated in Fig.1 for the potentials given by Eqs. (15) and (16) in terms of the fractional difference (17). This last factor represents the skewness of the non-Gaussian PDF. Consequently the sign of f_{NL} is what determines the enhancement or suppression of the probability for large amplitudes $\zeta_{\mathcal{R}}$ in our non-Gaussian PDF. Indeed, the non-Gaussian contribution encoded in the factor \mathcal{J} is the sum of the f_{NL} value over all scales relevant for PBH formation. For the two cases illustrated, the scalar tilt n_s dominates over the tensor tilt n_t in a way that the sign of f_{NL} incidentally coincides with that of n_s .

IV. CONSTRAINTS ON NON-GAUSSIAN PERTURBATIONS IN THE PBH RANGE

Primordial Black Holes are objects that collapsed from large-amplitude perturbations at times previous to photon decoupling [33]. The energy density of PBHs formed during inflation is diluted by the superluminous expansion. In consequence, a significant production of PBHs can only take place after inflation. Depending on the model of inflation, PBHs can cover a wide range of masses $M_{\text{END}} \sim 10^{-48} M_{\odot} \leq M_{\text{PBH}} \leq 10 M_{\odot}$ at scales much smaller than galaxies $M \sim 10^9 M_{\odot}$ or clusters $M \sim 10^{13} M_{\odot}$. We will call this the PBH mass range. The lack of direct observations of PBHs limits their cosmological abundance. Assuming they are nonetheless present, there are three ways in which PBHs can affect the evolution of our universe and conversely, we can impose constraints to the abundance of such objects. First, the current density of PBHs cannot exceed the amount of dark matter density, i.e., $\Omega_{\text{PBH}}(M \geq 10^{15} \text{g}) \leq \Omega_{\text{DM}} = 0.47$ (with the WMAP-III value taken at 3σ confidence level [1]). Second, the Hawking radiation [34] from PBHs, can also generate the radiation observed

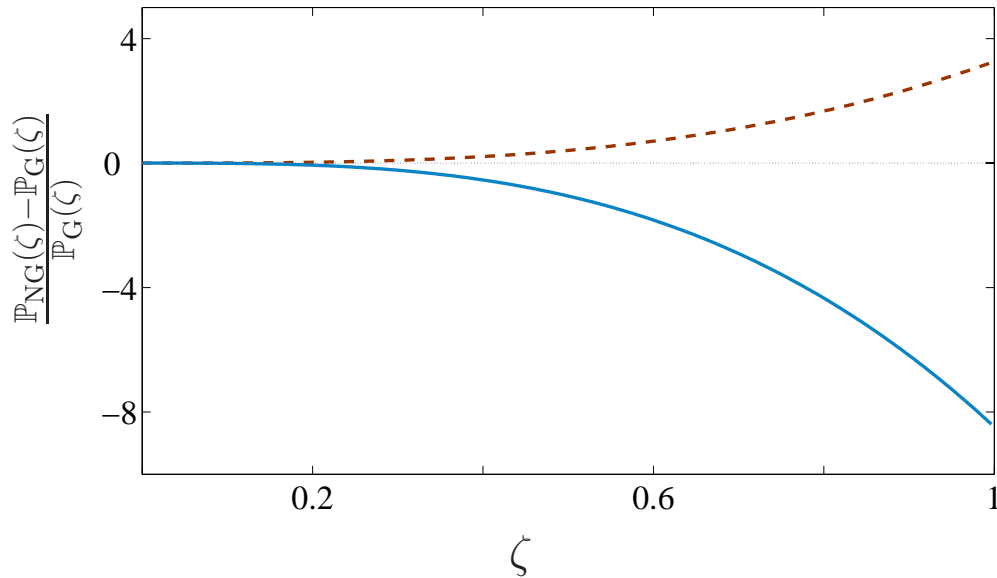


FIG. 1: The fractional departure from Gaussianity is plotted for two types of non-Gaussian distributions P_{NG} (10). For the potential in Eq. (15), $f_{\text{NL}} > 0$. The potential of Eq. (16) gives $f_{\text{NL}} < 0$ and its correspondent PDF is shown by a dashed line.

at various wavelengths in our universe [35]. The lifetime t_{evap} for PBHs is a function of its mass

$$t_{\text{evap}} = 1.2 \times 10^{-44} \left(\frac{M}{m_{\text{Pl}}} \right)^3 \text{ sec.} \quad (21)$$

From this relation we immediately infer that PBHs of mass $M_{\text{evap}} = 5 \times 10^{14} \text{g}$ are evaporating today and the observation of the gamma-ray background constrains their present density parameter to $\Omega_{\text{PBH}}(M_{\text{evap}}) \lesssim 5 \times 10^{-8}$ [35, 36, 37, 38]. This is the tightest constraint on the abundance of PBHs. A third category of constraints is relevant for lighter PBHs. Black holes with mass $M < M_{\text{evap}}$ have already evaporated and the decay products should not spoil the well understood chemical history of our universe (see e.g. Refs. [39, 40]).

To calculate the PBH mass fraction we make use of a standard Press-Schechter formalism [41]. This formula integrates the probability of PBH formation over the relevant matter perturbation amplitudes [33]. This is interpreted as the mass fraction of such PBHs at the time of formation,

$$\beta_{\text{PBH}}(\geq M) = 2 \int_{\delta_{\text{th}}}^{\infty} \mathbb{P}(\delta(M)) d\delta(M), \quad (22)$$

$$\approx \frac{\sigma_{\delta}(M)}{\delta_{\text{th}}} \exp \left[-\frac{\delta_{\text{th}}^2}{2\sigma_{\delta}^2(M)} \right]. \quad (23)$$

Here $\delta = \delta\rho/\rho$ is the matter density perturbation σ_{δ}^2 is the corresponding variance and δ_{th} is the threshold amplitude of the perturbation necessary to form a PBH. By integrating over a smoothed perturbation, this integral is equivalent to the mass fraction of PBHs of mass $M \geq \gamma^{3/2} M_H \approx \gamma^{3/2} k_M / (2\pi)$ [33], where γ is the sound-speed squared at the time formation. Note that the approximation (23) is valid only for a Gaussian PDF.

The integral (22) establishes a direct relation between the mass fraction of PBHs and the variance of perturbations. The set of observational constraints on the abundance of PBHs is listed in Table I and has been used to bound the mean amplitude of δ defined by for distinct cosmologies [31, 32, 42, 43]. The Press-Schechter formula has also been tested against other methods such as peaks theory [44].

The threshold value δ_{th} used in Eq. (22) has changed with the improvement of gravitational collapse studies [27, 33, 45, 46, 47]. Here we use the value $\delta_{\text{th}} = 0.3$ for convenience [57]. The corresponding threshold value of the curvature perturbation can be deduced from

[57] Recent studies suggest that this value is dependent on the profile of the curvature perturbation [48].

the relation [30]

$$\delta_k(t) = \frac{2(1+\gamma)}{5+3\gamma} \left(\frac{k}{aH} \right)^2 \mathcal{R}_k, \quad (24)$$

which at horizon crossing during the radiation dominated era gives $\mathcal{R}_{\text{th}} = 0.7$. Note that the high amplitude of the perturbations relevant to PBH formation necessarily require a non-linear treatment of their statistics. This is the major motivation for our analysis.

We adopt the Press-Schechter formula derive the non-Gaussian abundance of PBHs. The use of the Press-Schechter integral on distributions of curvature perturbations is not new. In Ref. [50] the Press-Schechter formula is used to integrate curvature perturbations which never exit the cosmological horizon. We apply the integral formula in Eq. (22) to the non-Gaussian probability distribution (10). The result of the integral is the sum of incomplete Gamma functions Γ_{inc} and an exponential:

$$\begin{aligned} \beta(M) = & \frac{1}{\sqrt{4\pi}} \Gamma_{\text{inc}} \left(1/2, \frac{\zeta_{\text{th}}^2}{2\Sigma_{\mathcal{R}}^2(M)} \right) - \\ & \frac{1}{\sqrt{2\pi}} \frac{\mathcal{J}}{\Sigma_{\mathcal{R}}^3(M)} \left[2 \Gamma_{\text{inc}} \left(2, \frac{\zeta_{\text{th}}^2}{2\Sigma_{\mathcal{R}}^2(M)} \right) - 3 \exp \left(-\frac{\zeta^2}{2\Sigma_{\mathcal{R}}^2(M)} \right) \right]. \end{aligned} \quad (25)$$

The Taylor series expansion of these functions around the limit $\Sigma_{\mathcal{R}}/\zeta_{\mathcal{R}} = 0$ gives

$$\beta(M) \approx \frac{\Sigma_{\mathcal{R}}(M)}{\zeta_{\text{th}} \sqrt{2\pi}} \exp \left[-\frac{1}{2} \frac{\zeta_{\text{th}}^2}{\Sigma_{\mathcal{R}}^2(M)} \right] \times \left\{ 1 - 2 \left(\frac{\Sigma_{\mathcal{R}}}{\zeta_{\text{th}}} \right)^2 + \frac{\mathcal{J}}{\Sigma_{\mathcal{R}}^3} \left[\left(\frac{\Sigma_{\mathcal{R}}}{\zeta_{\text{th}}} \right)^{-2} - 1 \right] \right\}. \quad (26)$$

For the mass fraction shown in Eq. (26), the observational limits of Table I could in principle constrain the values of the variance $\Sigma_{\mathcal{R}}^2$ and of f_{NL} . However, when we normalise the amplitude of perturbations to the observations at CMB scales, the limiting values for f_{NL} are too large, of order $f_{\text{NL}} \approx 10^4$, and thus inconsistent with perturbation theory at the level of the expansions performed in this paper. In fact, the expansion in Eq. (5) shows that when

$$|f_{\text{NL}}| \geq \frac{1}{\mathcal{R}^2} \approx \frac{1}{\mathcal{P}(k)} \approx 203.2, \quad (27)$$

the quadratic term of Eq. (5) dominates over the linear term, and in the computation of the three-point function Eq. (6), the loop contributions become dominant. The computation

Furthermore some early universe models may not generate the profiles required for PBH formation. This is a crucial aspect of PBH formation currently under investigation [49]

of non-Gaussianities in this case goes beyond the scope of this paper and should be treated carefully elsewhere. (for discussions on the origin and magnitude of the loop corrections see Refs. [51, 52, 53])

Here we restrict ourselves to use the marginal values allowed for f_{NL} from WMAP-III observations and look at the modifications that large non-Gaussianities bring to the amplitude of perturbations at the PBH scale.

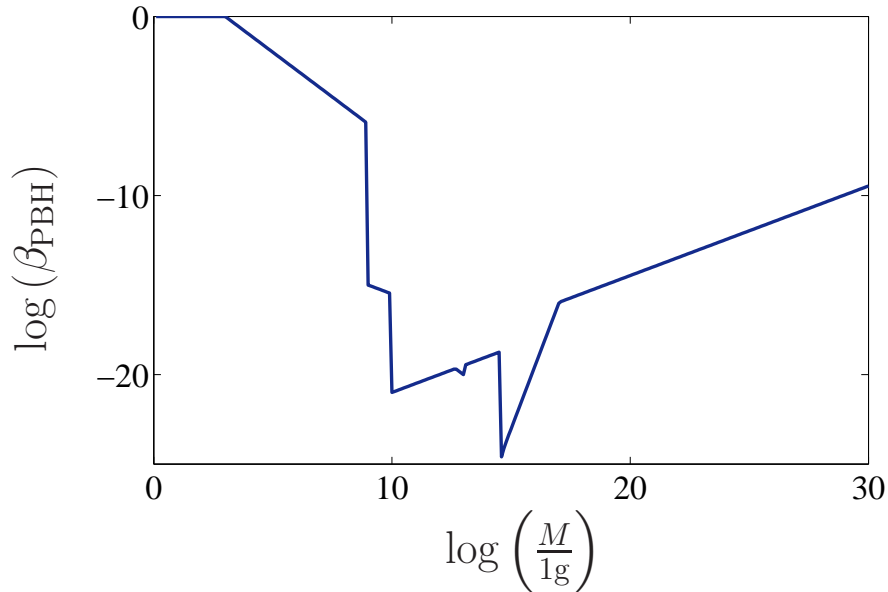


FIG. 2: The constraints in Table I are plotted together with only the smallest value considered for every mass.

In Fig. 2 we plot the set of bounds to the initial mass fraction of PBHs listed in Table I. The corresponding bounds on $\Sigma_{\mathcal{R}}$ are shown in Fig. 3 for the Gaussian and non-Gaussian cases. Independently of the model of cosmological perturbations adopted, one can use the observational limits of f_{NL} to modify the bounds for $\Sigma_{\mathcal{R}}$ on small wavelengths. The tightest constraints on $\Sigma_{\mathcal{R}}$ comes from perturbations of initial mass $M \approx 10^{15}\text{g}$. With the non-Gaussian modification the limit is $\log(\Sigma_{\mathcal{R}}) \leq -1.2$, compared to the Gaussian case $\log(\Sigma_{\mathcal{R}}) \leq -1.15$. As shown in Fig. 3, the modification to $\Sigma_{\mathcal{R}}$ cannot be much larger if we use the limit value of Eq. (27).

V. FINAL REMARKS

In this paper we have computed, to lowest order of non-linearity, the effects of non-Gaussian perturbations on PBHs formation.

We use curvature perturbations with a non-vanishing three-point correlation to find an explicit form of the non-Gaussian PDF with a direct contribution from the non-linear parameter f_{NL} . We have shown how the sign of this parameter determines the enhancement or suppression of probability for large-amplitude perturbations. Using the simple slow-roll expression for f_{NL} in the context of single field inflation, We have resolved previous discrepancies in the literature regarding effects of non-Gaussianity on the abundance of PBHs.

As a second application of the non-Gaussian PDF we have employed the Press-Schechter formalism of structure formation to determine the non-Gaussian effects on PBH abundance. We have shown how the PBH constraints on the amplitude of perturbations can be modified when a non-Gaussian distribution is considered. The maximum variance $\Sigma_{\mathcal{R}}$ allowed by

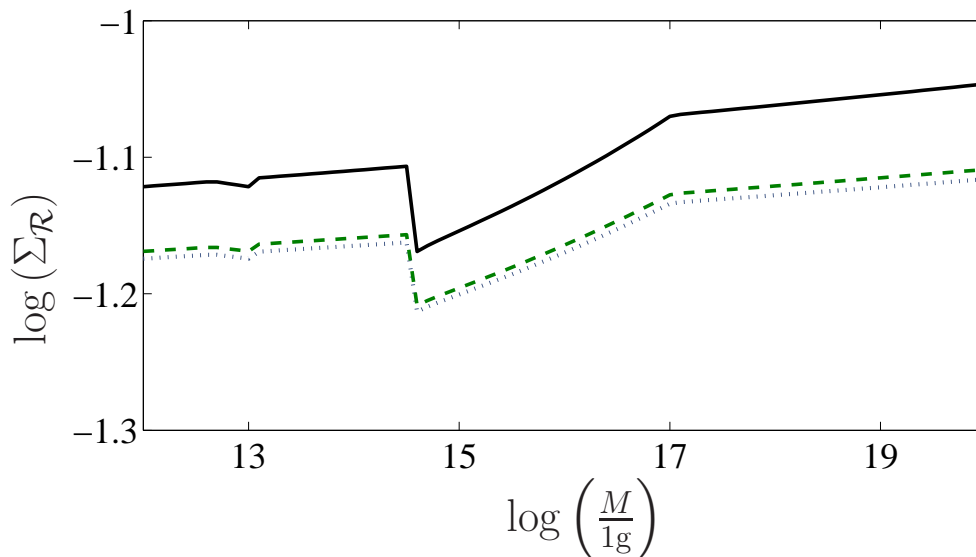


FIG. 3: A subset of the constraints on $\Sigma_{\mathcal{R}}$ from overproduction of PBHs is plotted for a Gaussian (black line) and non-Gaussian correspondence between β and $\Sigma_{\mathcal{R}}$, equations (23) and (26) respectively. The green dashed line assumes a constant $f_{\text{NL}} = -54$ and the blue dotted line a value $f_{\text{NL}} = -1/\Sigma_{\mathcal{R}}^2$.

PBH constraints is sensitive to f_{NL} , producing the limit $\Sigma_{\mathcal{R}}(M = 10^{15}\text{g}) < 6.3 \times 10^{-2}$ for $f_{\text{NL}} = -54$. This limit is, however, much larger than the observed amplitude at CMB scales, where $\Sigma_{\mathcal{R}} \approx 4.8 \times 10^{-5}$. The order of magnitude gap between the mean amplitude observed in cosmological scales and that required for significant PBH formation remains almost intact and, as a consequence, non-Gaussian perturbations do not modify significantly the standard picture of formation of PBHs.

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Table I Constraints on the fraction of PBH density to the density of the universe as described in
Ref. [54]

CONSTRAINT ($\beta \leq$)	MASS RANGE	NATURE
$1.5098 \times 10^{-8} (M/M_{\odot})^{1/2}$	$10^{15} - 10^{43} \text{g}$	$\Omega_{PBH}(\text{today}) \leq 1$
$2.1568 \times 10^{-16} \left(\frac{M}{M_{\odot}}\right)^{1/2} \times \left\{ 1 - \left[1 - \left(\frac{3.64 \times 10^{-9}}{M} \right)^3 \right]^{1/3} \right\}^{-1}$	$3.64 \times 10^{14} - 10^{15}$	X-rays from evaporating PBHs
$9.2 \times 10^{54} M^{11/2}$	$2.51 \times 10^{14} - 3.64 \times 10^{14}$	evaporated PBHs X-rays
$4.1 \times 10^{-3} \left(\frac{M}{10^9 \text{g}}\right)^{1/2}$	$10^9 - 10^{11}$	pair production at nucleosynthesis
$6.57 \times 10^{-5} \left(\frac{M}{10^{11} \text{g}}\right)^{7/2}$	$10^{11} - 10^{13}$	Helium-4 Spallation
$4.92 \times 10^{-7} \left(\frac{M}{10^{10} \text{g}}\right)^{3/2}$	$10^{10} - 10^{11}$	Deuterium destruction
$10^{-15} \left(\frac{M}{1 \times 10^9}\right)^{-1}$	$10^9 \text{g} - 10^{14} \text{g}$	CMB distortion
$10^{-18} \left(\frac{M}{1 \times 10^{11} \text{g}}\right)^{-1}$	$< 10^{11} \text{g}$	entropy of the universe